The weak maximum principle
Thursday, July 2nd 2015

Problem 1 Consider the control system

$$\ddot{x}(t) = \alpha \cos(u(t))$$
$$\ddot{y}(t) = \alpha \sin(u(t))$$

where $u(t) \in \mathbb{R}$ is the control. We want to minimize $t_f + g(x(t_f))$ where t_f is the final time and g is a $\mathcal{C}^1-$function. Find the optimal control.

Problem 2 Consider the control system

$$\dot{x}(t) = F_0(x(t)) + \sum_{i=0}^{m} u_i(t) F_i(x(t))$$

where the final time t_f is fixed, $t \in [0, t_f]$, $x(t)\ \mathbb{R}^n$, the control $u(t) = (u_1(t), \dots, u_m(t)) \in \mathbb{R}^m$ and $F_i, i = 0, \dots, m$ are smooth vector fields. We want to minimize the L^2-cost $\int_0^{t_f} |u|^2$. Find the extremal solutions of the problem in the normal case.

Problem 3 Let $F_i(l, x), i = 1, \dots, m$ be smooth vector fields parameterized by $l \in \mathbb{S}^1$ that set up a constant rank m distribution on $\mathbb{R}^n$. Consider a positive pulsation ω on $\mathbb{S}^1 \times \mathbb{R}^n$ relating the time t and the angle l according to

$$dl = \omega(l, x) dt. \tag{1}$$

We want to study the following problem

$$\frac{dx}{dl} = \sum_{i=1}^{m} u_i F_i(l, x),\ \ u \in \mathbb{R}^m,$$

$$\min_{u(.)} \int_0^{t_f} |u|^2 dt$$

The total angular length, $l_f > 0$ is fixed, implicitly defining t_f through (1).

1. Find the Hamiltonian $H_n(l, x, p)$ of which the extremal solutions of the control problem, in the normal case, are the integral curves.
2. Define the slow time $s = \epsilon l$ in $[0, 1]$, and renormalize the variables (x, p) by
$$\tilde{x} = x, \quad \tilde{p} = \frac{p}{\epsilon}.$$
Find the Hamiltonian system that describes the dynamics of $(\tilde{x}, \tilde{p})$ with respect to s.
3. Define the averaged Hamiltonian
$$H(z) = \frac{1}{2\pi} \int_0^{2\pi} H_n(l, z) dl.$$
It is well-known that the trajectories of the Hamiltonian system from the question 2. converges uniformly on $[0, 1]$ towards the solution of the averaged Hamiltonian (see, for instance, V.I. Arnold, Mathematical methods of classical mechanics, Springer, New-York, 1990]. Verify analytically that it is true for the simple control problem
$$\frac{dx}{dl} = \cos(l)u,$$
$$\min \int_0^{t_f} |u|^2 dt.$$
with some positive constant function $\omega(l, x)$.